Ergodic Theory and Measured Group Theory
Lecture 15

Theorem. For a pmp T on (X,μ), TFAE.

(1) T is weakly mixing, i.e. $\frac{1}{n+1}\sum_{i=0}^{n}\left|\langle T^i f, g\rangle - \int f \int g\right| \to 0$, $\forall f,g \in L^2(X,\mu)$.

(2) $\exists$ density 0 set $M \subseteq \mathbb{N}$, s.t. $\lim_{\substack{n\to\infty \\ n\notin M}} \langle T^i f, g\rangle = \int f \int g$.

(3) $T\times S$ on $(X\times Y, \mu\times\nu)$ is ergodic for every ergodic pmp S on (Y,ν).

(4) $T\times T$ on $(X\times X, \mu\times\mu)$ is ergodic.

(5) $T\times T$ on $(X\times X, \mu\times\mu)$ is weakly mixing.

(6) T is anticompact, i.e. $\forall$ nonconstant $f\in L^2(X,\mu)$, $\{T^i f : i\in\mathbb{N}\}$ is not precompact in $L^2(X,\mu)$.

Proof (continued). (1) $\Rightarrow$ (3). Enough to show that for any L^2-functions $f(x,y) := f_1(x)\cdot f_2(y)$, $g(x,y) := g_1(x)\cdot g_2(y)$, with $\int_{X\times Y} g \, d\mu\times\nu = \underbrace{\int_X g_1 \, d\mu}_{=: \bar g_1} \underbrace{\int_Y g_2 \, d\nu}_{=: \bar g_2} = 0$, the von Neumann ergodic theorem holds:

$$\frac{1}{n+1}\sum_{i=0}^{n} \langle (T\times S)^i f, g\rangle \xrightarrow[n\to\infty]{} 0 = \int_{X\times Y} f \, d\mu\times\nu \cdot \int_{X\times Y} g \, d\mu\times\nu.$$

$$\langle (T\times S)^i f, g\rangle = \underbrace{\langle (T\times S)^i f, \overbrace{(g_1 - \bar g_1)}^{g_1'} \times g_2\rangle}_{(1)} + \underbrace{\langle (T\times S)^i f, \bar g_1 \times g_2\rangle}_{(2)}$$

(1) $$\frac{1}{n+1}\sum_{i=0}^{n} \left|\langle (T\times S)^i f, g_1' \times g_2\rangle\right| \underset{\text{Fubini}}{=} \frac{1}{n+1}\sum_{i=0}^{n} \left|\langle T^i f_1, g_1'\rangle\right| \left|\langle S^i f_2, g_2\rangle\right|$$

$$[\text{Cauchy–Schwartz}] \leq \|f_2\|_2 \|g_2\|_2 \cdot \frac{1}{n+1} \sum_{i=0}^{n} |\langle T^i f_1, g_1' \rangle|$$

$$[\text{as } n \to \infty] \to 0 \quad \left(\text{recall } \int_X g_1' \, d\mu = 0\right).$$

The Cauchy-Schwartz step: $|\langle S^i f_2, g_2 \rangle| \overset{C-S}{\leq} \|S^i f_2\|_2 \cdot \|g_2\|_2$.
$= \|f_2\|_2 \|g_2\|_2$ because S^i is pmp so the change of variable holds.

(2) $\langle (T \times S)^i (f_1 \times f_2), \bar{g}_1 \times g_2 \rangle = \langle T^i f_1, \bar{g}_1 \rangle \langle S^i f_2, g_2 \rangle$

[$\bar{g}_1$ is constant and $= \bar{f}_1 \cdot \bar{g}_1 \cdot \langle S^i f_2, g_2 \rangle$
$\int T^i f_1 \, d\mu = \int f_1 \, d\mu =: \bar{f}_1$,
by change of variable].

$$\frac{1}{n+1} \sum_{i=0}^{n} \langle (T \times S)^i (f_1 \times f_2), \bar{g}_1 \times g_2 \rangle = \bar{f}_1 \cdot \bar{g}_1 \frac{1}{n+1} \sum_{i=0}^{n} \langle S^i f_2, g_2 \rangle$$

$$[\text{as } n \to \infty] \to \bar{f}_1 \cdot \bar{g}_1 \cdot \bar{f}_2 \cdot \bar{g}_2$$

$$[\bar{g}_1 \cdot \bar{g}_2 = 0] \quad = 0. \qquad \square$$

(3) $\Rightarrow$ (4). T on (X, μ) is in particular ergodic, so take $S := T$ & $Y := X$.

(4) $\Rightarrow$ (1). Let $f, g \in L^2(X, \mu)$ & we may assume that $g \in L^2_0(X, \mu) :=$

$\{h \in L^2(X,\mu) : \int h \, d\mu = 0\}$. We want to show

$$\frac{1}{n+1}\sum_{i=0}^{n} |\langle T^i f, g\rangle| \xrightarrow[n\to\infty]{} 0 = \int f\,d\mu \int g\,d\mu.$$

We apply the von Neumann ergodic thm to $f\times f$ and $g\times g$:

$$\frac{1}{n+1}\sum_{i=0}^{n} \underbrace{\langle (T\times T)^i (f\times f), g\times g\rangle}_{= |\langle T^i f, g\rangle|^2} \to \underbrace{\int f\times f \, d\mu\times\mu \cdot \int g\times g \, d\mu\times\mu}_{= \left(\int f\,d\mu\right)^2 \cdot \left(\int g\,d\mu\right)^2 = 0}$$

By Cauchy-Schwartz in $\mathbb{R}^{n+1}$, $\left(\frac{1}{n+1}\sum_{i=0}^{n} |\langle T^i f, g\rangle|\right)^2 \leq \frac{1}{n+1}\sum_{i=0}^{n} |\langle T^i f, g\rangle|^2 \xrightarrow[n\to\infty]{} 0$.

(1) => (5). We know that weak mixing of $T\times T$ is equivalent to the ergodicity of $T\times T\times S$ for any pmp ergodic S. But if T is weak mixing, then $T\times S$ is ergodic and pmp, and hence $T\times(T\times S)$ is ergodic.

(5) => (4). Trivial.

(1) => (6). Enough to show that $\forall f \in L^2_0(X,\mu)$, if $\{T^i f : i \in \mathbb{N}\}$ is precompact, then $f = 0$. Suppose $\{T^i f : i\in\mathbb{N}\}$ is precompact, hence has finite ε nets. But weak mixing makes

$T^n f$ orthogonal from every vector, eventually, so this should be a contradiction. Let $\varepsilon < \frac{1}{2}\|f\|_2$, and let $\{f_1, f_2, \dots, f_m\}$ be an ε-net for $\{T^i f : i \in \mathbb{N}\}$. Thus, using T is mp,

$$\|f\|_2^2 = \|T^i f\|_2^2 = \langle T^i f, f_k \rangle + \langle T^i f, T^i f - f_k \rangle$$

[change of var] choose k so that $\|T^i f - f_k\|_2 < \varepsilon$.

$$[\text{Cauchy-Schwartz}] \leq |\langle T^i f, f_k \rangle| + \|T^i f\|_2 \cdot \|T^i f - f_k\|_2$$

$$[\text{change of var}] = |\langle T^i f, f_k \rangle| + \|f\|_2 \cdot \varepsilon$$

$$\leq \sum_{k=1}^{m} |\langle T^i f, f_k \rangle| + \|f\|_2 \cdot \varepsilon .$$

Taking averages: $$\|f\|_2^2 \leq \frac{1}{n+1} \sum_{i=0}^{n} \sum_{k=1}^{m} |\langle T^i f, f_k \rangle| + \|f\|_2 \cdot \varepsilon$$

$$[\text{baby Fubini}] = \sum_{k=1}^{m} \frac{1}{n+1} \sum_{i=0}^{n} |\langle T^i f, f_k \rangle| + \|f\|_2 \cdot \varepsilon$$

$$[n \to \infty] \to 0 + \|f\|_2 \cdot \varepsilon < \frac{1}{2}\|f\|_2 ,$$

hence $f = 0$.

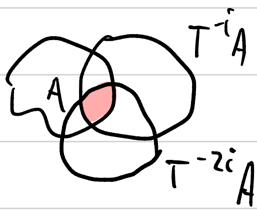
(6) => (1). This is a bit too involved, read it in Joel Moreira's blog, topic: weak mixing. $\square$

Weak mixing is also equivalent to a strong form of multiple recurrence which we state now. This is 50% of why the famous Multiple Recurrence Theorem of Furstenberg holds (this theorem implies Szemeredi's theorem).

Theorem* (van der Corput / Furstenberg). An invertible mpp T is weakly mixing if and only if $\forall k \in \mathbb{N}$

$$\frac{1}{n+1}\sum_{i=0}^{n} \mu(A \cap T^{-i}A \cap T^{-2i}A \cap \dots \cap T^{-ki}A) \to \mu(A)^k$$

$\forall^\infty i$ out of a density 0 set



has measure $\mu(A)\cdot\mu(T^{-i}A)\cdot\mu(T^{-2i}A) = \mu(A)^3$.

Corollary (Furstenberg's Multiple Recurrence for weak mixing T). $\forall$ weakly mixing mpp T on (X,μ) and $A \subseteq X$ of positive measure, $\forall k\ \exists\, n \geq 1$ s.t.

$$\mu(A \cap T^{-n}A \cap T^{-2n}A \cap \dots \cap T^{-kn}A) > 0.$$

Proof of Theorem*. $\Leftarrow$. Too hard, will try to find a good ref.

$\Rightarrow$ (the important direction towards Multiple Rec.).

Read the proof in Joel Moreira's blog.
But it uses the following famous trick:

Lemma (van der Corput). Let $(u_n)_{n\in\mathbb{N}}$ be a bounded sequence of vectors in a Hilbert space H.
If the u_n are very weakly pairwise orthogonal, then $\frac{1}{n+1}\sum_{i=0}^{n} u_i \to 0$ as $n\to\infty$.

!!

$$\lim_{m\to\infty}\sum_{s=0}^{m}\limsup_{n\to\infty}\frac{1}{n+1}\sum_{i=0}^{n}\langle u_i, u_{i+s}\rangle = 0.$$

This lemma amplifies weak mixing (=: 1-recurrence) to multiple recurrence.